MMP Learning Seminar

Week 48.

Content:

- Base point free Theorem.
- Termination of flips.

Base point free Theorem:

Proposition: Let (X, Δ) be a dlt pair and H be a nef $\mathbb{Q}$-Cartier divisor such that $H - (K_X + \Delta)$ is nef and abundant. Assume that

$\nu(X, aH - (K_X + \Delta)) = \nu(X, H - (K_X + \Delta))$ &

$\kappa(X, aH - (K_X + \Delta)) \geq 0$ for some $a > 1$.

If $H|_{\lfloor \Delta \rfloor}$ is semiample, then $\mathrm{Bs}\,|mH| \cap \lfloor \Delta \rfloor = \emptyset$ for some m with mH Cartier.

Proof: $X \xleftarrow{\mu} Y \xrightarrow{f} Z$, μ is a log resolution & f is a contraction.

$$\mu^*(H - (K_X + \Delta)) \sim_{\mathbb{Q}} f^* M_0 , \quad M_0 \text{ big \& nef}$$

$$\mu^* H \sim_{\mathbb{Q}} f^* H_0 , \quad \text{with } H_0 \text{ nef}$$

$$K_Y = \mu^*(K_X + \Delta) + \sum a_i E_i , \quad a_i \geq -1.$$

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$$K_Y = \mu^*(K_X + \Delta) + \sum a_i E_i, \quad a_i \geq -1.$$

$$S := \lfloor \Delta \rfloor, \quad E := \sum_{a_i > 0} \lceil a_i \rceil E_i, \quad S' := \sum_{a_i = -1} E_i.$$

Note that

$$m\mu^* H + E - S' - (K_Y + \underbrace{\textstyle\sum \{-a_i\} E_i}_{\text{klt}}) =$$

$$(m-1)\mu^* H + \mu^*(H - (K_X + \Delta)) \sim_{\mathbb{Q}}$$

$$(m-1) f^* H_0 + f^* M_0$$

nef + big on Z.

Case 1.- $f(S') \neq Z$. By Kollar's injectivity Theorem:

$$H^1(Y, \mathcal{O}_Y(m\mu^* H + E - S')) \longrightarrow$$

$$H^1(Y, \mathcal{O}_Y(m\mu^* H + E)) \quad \text{is injective.}$$

Then, we have a commutative diagram:

$$
\begin{tikzcd}
\beta \neq 0 & \alpha \neq 0 \\
H^0(Y, \mathcal{O}_Y(m\mu^* H + E)) \arrow[r, "\text{surj}"] & H^0(S', \mathcal{O}_{S'}(m\mu^* H + E)) \\
H^0(Y, \mathcal{O}_Y(m\mu^* H)) \arrow[u, "s"] \arrow[r] & H^0(S', \mathcal{O}_{S'}(m\mu^* H)) \arrow[u, "i"] \\
H^0(X, \mathcal{O}_X(mH)) \arrow[u, "s"] \arrow[r, "s"] & H^0(S, \mathcal{O}_S(mH)) \arrow[u, "j"] \\
\beta' \neq 0 \arrow[r, mapsto] & \alpha \neq 0
\end{tikzcd}
$$

i is injective because S' & E have no common comp.

j is injective because $S' \longrightarrow S$ is surjective

Hence, s is surjective. Thus, $|mH|_S = |mH|_S| = \emptyset$.

Case 2: $f(S') = Z$.

There exists S'' of S' with $f(S'') = Z$.

Since $H|_S$ semiample and $\mu^* H \sim_{\mathbb{Q}} f^* H_0$, then $f^* H_0|_{S''}$ is semiample. Hence, H_0 is semiample. $\square$

Proposition: Assume (X, Δ) is dlt. H nef $\mathbb{Q}$-div on X such that:

(1) $H - (K_X + \Delta)$ is nef & abundant,

(2) $\nu(X, aH - (K_X+\Delta)) = \nu(X, H - (K_X + \Delta))$, and

$\kappa(X, aH - (K_X + \Delta)) \geq 0$.

If for some $p_1 \in \mathbb{Z}_{\geq 1}$, the divisor $p_1 H$ is Cartier and $\mathrm{Bs}|p_1 H| \cap \lfloor \Delta \rfloor = \phi$, then H is semiample

Theorem: (X, B) lc pair & $\pi : X \longrightarrow S$ proper morphism onto S.

Assume the following conditions:

(a) H is π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X,

(b) $H - (K_X + B)$ is π-nef & π-abundant,

(c) $\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and

$\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$

for some $a > 1$.

(d) cH is Cartier and $\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$ is π-generated,

where $T = \mathrm{Nonklt}(X, B)$.

Then H is π-semiample.

Theorem 4.1: $f: X \longrightarrow U$ projective.

(X, Δ) dlt $\mathbb{Q}$-factorial. $U^\circ \subseteq U$ open :

(1) The image of any strata S_I of $S = \lfloor \Delta \rfloor$ intersects U°,

(2) $K_X + \Delta$ is π-nef and $(K_X + \Delta)|_{X^\circ}$ is semiample

(3) for any component S_i of S, $(K_X + \Delta)|_{S_i}$ is semiample over U.

Then $K_X + \Delta$ is semiample over U.

Theorem 1.1: $f: X \longrightarrow U$ proj mor of normal var.
(X, Δ) dlt, $S = \lfloor \Delta \rfloor$, $U^\circ \subseteq U$ is such that
(X°, Δ°) has a good minimal model over U°.
Assume every strata of S intersects X°.
Then, (X, Δ) has a good minimal model over U.

Remark: $\mu^*(K_X + \Delta) + F = K_{X'} + \Delta'$, then

(X', Δ') has a gmm over $U \Longrightarrow (X, \Delta)$ has a gmm over

$$
\begin{tikzcd}
 & X^\circ \arrow[dl, dashed] \arrow[d] \arrow[r, phantom, "\subseteq"] & X \arrow[d] \\
\overline{Y}^\circ \arrow[r] & U^\circ \arrow[r, phantom, "\subseteq"] & U
\end{tikzcd}
$$

(X°, Δ°) has a gmm over U°.

every strata of $S = \lfloor \Delta \rfloor$ intersects X°

(X, Δ) is dlt

$$
\begin{tikzcd}
 & X^\circ \arrow[dl, dashed] \arrow[dd] & \subseteq & X \arrow[dd] \\
\bar{Y}^\circ \arrow[rr, hook] \arrow[dr] & & \bar{Y} \arrow[dr] & \\
 & U^\circ & \subseteq & U
\end{tikzcd}
$$

Outcome of 5.1:

(Z, Δ_Z) is a min model for (X, Δ) over Y.

$$\begin{tikzcd}
 & X^\circ \arrow[dl, dashed] \arrow[dd] & \subseteq & X \arrow[dd] \arrow[dr, "h_X"] \arrow[rr, dashed, "\psi"] & & Z \arrow[dl, "h"] \\
\overline{Y}^\circ \arrow[rr, hook] \arrow[dr] & & \overline{Y} \arrow[dr] & & Y \arrow[ll] \arrow[dl] & \\
 & U^\circ & \subseteq & U & &
\end{tikzcd}$$

$(Y, B+J)$

$(Y, B+J+\varepsilon(A+C))$

$\sim_{\mathbb{Q}}$

(Y, Θ_ε).

Outcome of 5.2 + 5.3:

$$\begin{tikzcd}
 & X^\circ \arrow[dl, dashed] \arrow[dd] & \subseteq & X \arrow[dr, "h_X"] \arrow[dd] \arrow[rr, dashed, "\psi"] \arrow[rrrr, dashed, bend left, "\psi'"] & & Z \arrow[dl, "h"'] \arrow[rr, dashed] & & Z' \arrow[dl, "h'"] \\
\bar{Y}^\circ \arrow[rrr, hook] \arrow[dr] & & & \bar{Y} \arrow[d] & Y \arrow[l] \arrow[dl] \arrow[rr, dashed] & & Y' \arrow[dlll] & \\
 & U^\circ & \subseteq & U & & & &
\end{tikzcd}$$

$K_X+\Delta$ is ~~nef~~ semiample over U $\Longrightarrow$ ample model

$$\begin{tikzcd}
X \arrow[d] & K_X+\Delta \equiv 0 \text{ on the fiber} \\
Y_D \arrow[d] & K_{Y_D} \text{ ample} \\
U &
\end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, dashed, "\psi'"] \arrow[d, "f"'] & Z' \arrow[d, "h'"] \\ U & Y' \arrow[l] \end{tikzcd}$$

- $K_{Y'}+B'+J'+t(C'+A')$ semiample over U^0 for every $0 \le t \le \varepsilon$
- $\underline{K_{Y'}+B'+J'+\varepsilon(C'+A')}$ semiample over U.
- $K_X+\Delta+\varepsilon h_X^*(C+A)$ has a min model Z' over U.

Aim: produce a min model for $K_{Y'}+B'+J'$ over U.

"Natural": Run a $(K_{Y'}+B'+J')$-MMP with scaling of $C'+A'$ over U.

Lemma 5.1: We can find a proj bir morphism $Y \longrightarrow \overline{Y}$ a morphism $h_X : X \longrightarrow Y$. :

(1) (X, Δ) has a good minimal model Z over Y,

(2) $h_{X*}(\mathcal{O}_X(m(K_X + \Delta)) \simeq \mathcal{O}_Y(m(K_Y + B + J))$

(3) $K_Z + \psi_* \Delta \sim_{\mathbb{Q}} h^*(K_Y + B + J)$

(4) (Y, B) is log smooth and all strata of $T = \lfloor B \rfloor$ intersects Y°.

(5) $K_Y + B + J \sim_{\mathbb{Q}, U} C + \Sigma + A$, $(Y, B + \varepsilon C)$, $(X, \Delta + \varepsilon h_X^* C)$ are dlt

(6) for any $0 < \varepsilon \ll 1$, there exists $\Theta_\varepsilon \sim_{\mathbb{Q}, U} B + J + \varepsilon(A + C)$ so that (Y, Θ_ε) is klt

Proof of 5.1: Proposition 2.1 $(h_X : X \longrightarrow Y, B, J)$

$$K_Y + B + J \sim_{\mathbb{Q},U} C + \Sigma' + A. \qquad T = \lfloor B \rfloor$$

A ample, Σ' & C have no common comp, $\mathrm{Supp}(\Sigma') \subseteq \mathrm{Supp}(T)$

Assume $(Y, \mathrm{supp}(B+C))$ is log smooth

C contains no non-klt centers of $(Y, B) \implies$

$h_X^* C$ contains no non-klt centers of (X, Δ).

Observe $J + \varepsilon A$ ample over U.

Hence, $B + J + \varepsilon(C+A) \sim_{\mathbb{Q},U} \Theta_\varepsilon$ where (Y, Θ_ε) kl

$$\underbrace{B - \delta T + \varepsilon C}_{\text{klt}} + \underbrace{J + \varepsilon A + \delta T}_{\text{ample}} = B + J + \varepsilon(C+A).$$

Proposition 2.13, gives us an isomorphism:

$$R(X/Y, d(K_X + \Delta + \varepsilon h_X^*(C+A))) \cong$$

$$R(Y/Y, d(K_Y + B + J + \varepsilon(C+A))) \cong$$

$$R(Y/Y, d(K_Y + \Theta_\varepsilon)) \longleftarrow \text{is f.g.}$$

By proposition 2.11 + 2.12. $\Longrightarrow$ (X, Δ) has a good minimal model (Z, Δ_Z) for (X, Δ) over Y

As an outcome, we conclude that

$$K_Z + \Delta_Z \sim_{\mathbb{Q},Y} 0 \quad \text{so} \quad K_Z + \Delta_Z \sim_{\mathbb{Q}} h^*(K_Y + B + J).$$

$\square$

Lemma 5.2 + 5.3: Up to replacing X with a higher model:

(1) $K_Y + B + J + \varepsilon(C + A)$ has a gmm Y' over U.

(2) $K_X + \Delta + \varepsilon h_X^*(C+A)$ has a gmm Z' over U,

equipped with a morphism $Z' \xrightarrow{h'} Y'$.

(3) $(K_{Y'} + B' + J' + t(C' + A'))|_{Y^0}$ is semiample over U^0.

for every $0 \leq t \leq \varepsilon$

(4) $N_\sigma(K_Y + B + J + \varepsilon(C+A)/U)$ stabilize for ε small

Prop 2.13.

$(K_X+\Delta)$-MMP with scaling of $h_X^*(C+A)$.

$$
\begin{tikzcd}
X \arrow[r, dashed] \arrow[dd] & Z' \arrow[r, dashed] \arrow[d, "h'"] & Z'_1 \arrow[r, dashed] \arrow[d] & \cdots \arrow[r, dashed] & Z'_i \arrow[d] \\
 & Y' \arrow[r, dashed] \arrow[dl] & Y'_1 \arrow[r, dashed] \arrow[dll] & \cdots \arrow[r, dashed] & Y'_i \arrow[dllll, bend left] \\
U & & & &
\end{tikzcd}
$$

$K_{Y'}+B'+J'$-MMP with scaling of $C'+A'$

$(K_X+\Delta)$-MMP with scaling of $h_X^*(C+A)$.

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[ddd] & Z' \arrow[r, dashed] \arrow[d, "h'"] & Z'_1 \arrow[r, dashed] \arrow[d] & \cdots \arrow[r, dashed] & Z'_i \arrow[d] \\
& Y' \arrow[r, dashed] \arrow[ddl] & Y'_1 \arrow[r, dashed] \arrow[ddll] & \cdots \arrow[r, dashed] & Y'_i \arrow[ddllll, bend left] \\
& & & & \\
U & & & &
\end{tikzcd}$$

$K_{Y'}+B'+J'$-MMP with scaling of $C'+A'$

Claim: This MMP terminates.

$Y' \dashrightarrow Y'_N$, we obtain $Z \dashrightarrow Z'_N$ a min model for $K_Z+\Delta_Z+th^*(C+A)$ and

$$R(X/U, K_X+\Delta) \cong R(Z'_N/U, K_{Z'_N}+\Delta_{Z'_N}).$$

$$R(Z^\circ_N/U^\circ, K_{Z^\circ_N}+\Delta_{Z^\circ_N}) \cong R(X^\circ_N/U^\circ, K_{X^\circ}+\Delta^\circ) \text{ is f.g.}$$

By induction on dimension, $K_{T'_N}+\Delta_{T'_N} = (K_{Z'_N}+\Delta_{Z'_N})|_{T'_N}$ is semiample over U.

4.1 $\Longrightarrow$ $K_{Z'_N}+\Delta_{Z'_N}$ is semiample over U. $\square$

Termination of flips:

Notation: D_i' push-forward of D in Y_i'.

$(K_{Y'} + B' + J')$ - MMP over U with scaling of $\varepsilon(C'+A')$

$T_i = \lfloor B_i' \rfloor$ for every i.

Γ_i is a flipping curve. $(C_i' + A_i')\cdot\Gamma_i > 0$

$(K_{Y_i'} + J_i' + B_i')\cdot\Gamma_i < 0$

so $\Sigma_i \cdot \Gamma_i < 0$.

$\mathrm{supp}(\Sigma_i) = \mathrm{supp}(\lfloor B_i \rfloor) = \mathrm{supp}(T_i)$

The whole MMP happens along non-klt (Y', B').

G_i' a component of $S = \lfloor \Delta \rfloor$ dominating a component of T

We obtain a MMP $G_i' \dashrightarrow G_{in}$ for the pair (G_i, Δ_{G_i}) obtained by adjunction with scaling of $\Phi_i = h_i^*(A_i' + C_i')|_{G_i}$.

$$
\begin{tikzcd}
 & G' \arrow[r, dashed] \arrow[d, phantom, "\cap"] & G_1' \arrow[r, dashed] \arrow[d, phantom, "\cap"] & \cdots \arrow[r, dashed] & G_i' \arrow[d, phantom, "\cap"] \\
X \arrow[r, dashed] \arrow[dd] & Z' \arrow[r, dashed] \arrow[d, "h'"] & Z_1' \arrow[r, dashed] \arrow[d] & \cdots \arrow[r, dashed] & Z_i' \arrow[d] \\
 & Y' \arrow[r, dashed] \arrow[dl] & Y_1' \arrow[r, dashed] \arrow[dll] & \cdots \arrow[r, dashed] & Y_i' \arrow[dllll, bend left] \\
U & & & &
\end{tikzcd}
$$

This MMP terminates.

$\dim G_i < \dim Z_i$

eventually disjoint from the G_i's.

By Thm 1.1 dim $n-1$, $K_{G_i} + \Delta_{G_i}$ has a gmm $\bar{G}$ over U.

$\chi : \bar{G} \longrightarrow \bar{D}$ ample model over U.

Lemma 5.7: There is a gmm $\bar{G}^m$ of $(\bar{G}, \Delta_{\bar{G}} + t_0 \bar{\Phi})$ over $\bar{D}$.

Proposition: $(\bar{G}_m, \Delta_{\bar{G}_m} + t \bar{\Phi}_m)$ is a gmm over U for all $0 \leq t \leq t_0$. Hence, the multi-graded ring

$R(\bar{G}_m / U, K_{\bar{G}_m} + \Delta_{\bar{G}_m} ; K_{\bar{G}_m} + \Delta_{\bar{G}_m} + t_0 \bar{\Phi}_m)$ is f.g.

Since $\bar{G} \dashrightarrow \bar{G}_m$ is $K_{\bar{G}} + \Delta_{\bar{G}} + t \bar{\Phi}$ non-pos

$R(\bar{G} / U, K_{\bar{G}} + \Delta_{\bar{G}} ; K_{\bar{G}} + \Delta_{\bar{G}} + t_0 \bar{\Phi})$ is also f.g.

Lemma 5.8: The $Z_i \dashrightarrow Z_{i+1}$ are eventually disjoint from G_i.

Proof: $\mathcal{I}$ geom val over $\overline{G}$. $\sigma_{\mathcal{I}}(D) = \inf\{ \operatorname{mult}_{\mathcal{I}}(D') \mid D' \sim D \}$

$$\sigma_{\mathcal{I}}(K_{\overline{G}^m} + \Delta_{\overline{G}^m} + t\,\overline{\Phi}^m / U) = 0 \quad \text{for all } t \in [0, t_0]$$

$\sigma_{\mathcal{I}}(\cdot / U)$ is ≥ 0 and linear on the cone $\mathcal{C}_i \subseteq \operatorname{Div}(G_i)$ spanned by $\gamma_i^*(K_{G_i} + \Delta_{G_i})$ and $\gamma_i^*(K_{G_i} + \Delta_{G_i} + t_0 \Phi_i)$. [CL10].

There exists E over G_i' s.t. for every $0 < \delta < s_i$ $\nearrow$ scaling multiple

$\alpha(E_i, G_i, K_{G_i} + \Delta_{G_i} + (s_i - \delta)\Phi_i) <$

$\alpha(E_1, G_{i+1}, K_{G_{i+1}} + \Delta_{G_{i+1}} + (s_{i+1} - \delta)\Phi_{i+1})$.

This implies $\sigma_{E_i}(K_{G_i} + \Delta_{G_i} + (s_i - \delta)\Phi_i / U) > 0$.

$$\sigma_E(K_{G_i} + \Delta_{G_i} + s_i \Phi_i / U) = 0.$$

$\rightarrow\leftarrow$ σ_E is linear and non-neg on $\mathcal{C}_i$. $\square$