

MMP Learning Seminar

Week 48.

Content:

- Base point free Theorem.
- Termination of flips.

Base point free Theorem:

Proposition: Let (X, Δ) be a dlt pair and H be a nef $\mathbb{Q}$ -Cartier divisor such that $H - (K_X + \Delta)$ is nef and abundant. Assume that

$$\gamma(X, \alpha H - (K_X + \Delta)) = \gamma(X, H - (K_X + \Delta)) \quad \& \\ \kappa(X, \alpha H - (K_X + \Delta)) \geq 0 \quad \text{for some } \alpha > 1.$$

If $H|_{\Delta}$ is semiample, then $\text{Bs} |mH| \cap \Delta = \emptyset$ for some m with mH Cartier.

Proof: $X \xleftarrow{\mu} Y \xrightarrow{f} Z$, μ is a log resolution
& f is a contraction.

$$\mu^*(H - (K_X + \Delta)) \sim_{\mathbb{Q}} f^* M_0, \quad M_0 \text{ big \& nef}$$

$$\mu^* H \sim_{\mathbb{Q}} f^* H_0, \quad \text{with } H_0 \text{ nef}$$

$$K_Y = \mu^*(K_X + \Delta) + \sum a_i E_i, \quad a_i \geq -1.$$

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$$K_Y = \mu^*(K_X + \Delta) + \sum_i \alpha_i E_i, \quad \alpha_i \geq -1.$$

$$S := \lfloor \Delta \rfloor, \quad E := \sum_{\alpha_i > 0} \lfloor \alpha_i \rfloor E_i, \quad S' := \sum_{\alpha_i = -1} E_i.$$

Note that

$$m \mu^* H + E - S' - (K_Y + \overset{\text{KLT}}{\sum_i \{-\alpha_i\} E_i}) =$$

$$(m-1) \mu^* H + \mu^*(H - (K_X + \Delta)) \sim_{\mathbb{Q}}$$

$$(m-1) f^* H_0 + f^* M_0$$

$\rightarrow$ nef + big on Z .

Case 1.- $f(S') \neq Z$. By Kollar's injectivity Theorem:

$$H^1(Y, \mathcal{O}_Y(m \mu^* H + E - S')) \longrightarrow$$

$$H^1(Y, \mathcal{O}_Y(m \mu^* H + E)) \quad \text{is injective.}$$

Then, we have a commutative diagram

$$\begin{array}{ccc}
 H^0(Y, \mathcal{O}_Y(m\mu^*H + E)) & \xrightarrow{\text{surj}} & H^0(S', \mathcal{O}_{S'}(m\mu^*H + E)) \\
 \uparrow s & & \uparrow i \\
 H^0(Y, \mathcal{O}_Y(m\mu^*H)) & \longrightarrow & H^0(S', \mathcal{O}_{S'}(m\mu^*H)) \\
 \uparrow s & & \uparrow j \\
 H^0(X, \mathcal{O}_X(mH)) & \xrightarrow{s} & H^0(S, \mathcal{O}_S(mH))
 \end{array}$$

$\beta \neq 0$ $\alpha \neq 0$
 $\beta' \neq 0$ $\alpha' \neq 0$

i is injective because S' & E have no common comp.

j is injective because $S' \rightarrow S$ is surjective

Hence, s is surjective. Thus, $|mH|_S = |mH|_{S'} = \emptyset$.

Case 2: $f(S') = \mathbb{Z}$.

There exists S'' of S' with $f(S'') = \mathbb{Z}$.

Since $H|_S$ semiample and $\mu^*H \sim f^*H_0$, then

$f^*H_0|_{S''}$ is semiample. Hence, H_0 is semiample. $\square$

Proposition: Assume (X, Δ) is dlt. H nef $\mathbb{Q}$ -div on X such that:

- (1) $H - (K_X + \Delta)$ is nef & abundant.
- (2) $\nu(X, \alpha H - (K_X + \Delta)) = \nu(X, H - (K_X + \Delta))$, and $\kappa(X, \alpha H - (K_X + \Delta)) \geq 0$.

If for some $p_i \in \mathbb{Z}_{\geq 1}$, the divisor $p_i H$ is Cartier and $B \cap p_i H \cap \lfloor \Delta \rfloor = \emptyset$, then H is semiample.

Theorem: (X, B) lc pair & $\pi: X \rightarrow S$ proper morphism onto S .

Assume the following conditions:

- (a) H is π -nef $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on X ,
- (b) $H - (K_X + B)$ is π -nef & π -abundant,
- (c) $\kappa(X_\eta, (\alpha H - (K_X + B))_\eta) \geq 0$ and

$$\nu(X_\eta, (\alpha H - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$$

for some $\alpha > 1$.

- (d) cH is Cartier and $\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$ is π -generated,

where $T = \text{Nonklt}(X, B)$.

Then H is π -semiample.

Theorem 4.1: $f: X \rightarrow U$ projective.

(X, Δ) dlt $\mathbb{Q}$ -factorial. $U^\circ \subseteq U$ open:

- (1) The image of any strata S_i of $S = \lfloor \Delta \rfloor$ intersects U° .
- (2) $K_X + \Delta$ is $\mathbb{R}$ -nef and $(K_X + \Delta)|_{X^\circ}$ is semiample.
- (3) for any component S_i of S , $(K_X + \Delta)|_{S_i}$ is semiample over U .

Then $K_X + \Delta$ is semiample over U .

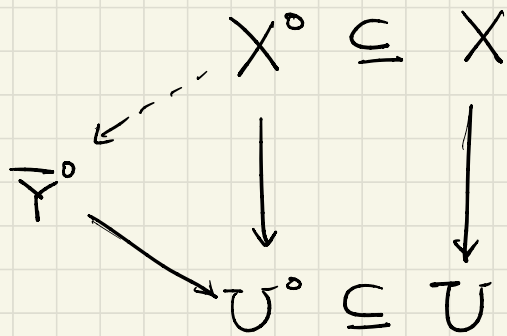
Theorem 1.1: $f: X \rightarrow U$ proj mor of normal var.
 (X, Δ) dlt, $S = \lfloor \Delta \rfloor$, $U^0 \in U$ is such that
 (X^0, Δ^0) has a good minimal model over U^0 .

Assume every strata of S intersects X^0 .

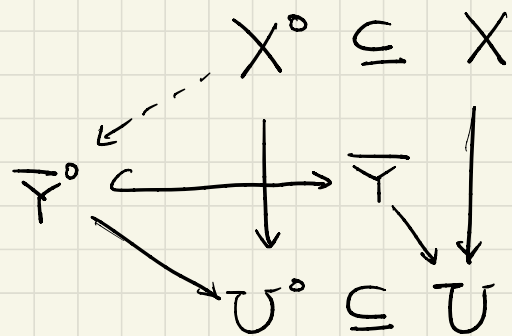
Then, (X, Δ) has a good minimal model over U .

Remark: $\mu^*(K_X + \Delta) + F = K_{X'} + \Delta'$, then

(X', Δ') has a gmm over $U \implies (X, \Delta)$ has a gmm over

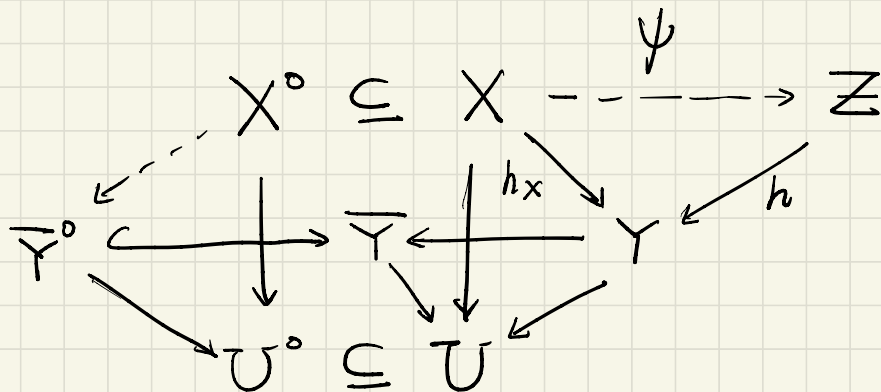


(X^o, Δ^o) has a gmm over U^o .
 every strata of $S = L\Delta$ intersect X^o
 (X, Δ) is dlt



Outcome of 5.1:

(Z, Δ_Z) is a min model
for (X, Δ) over Y .



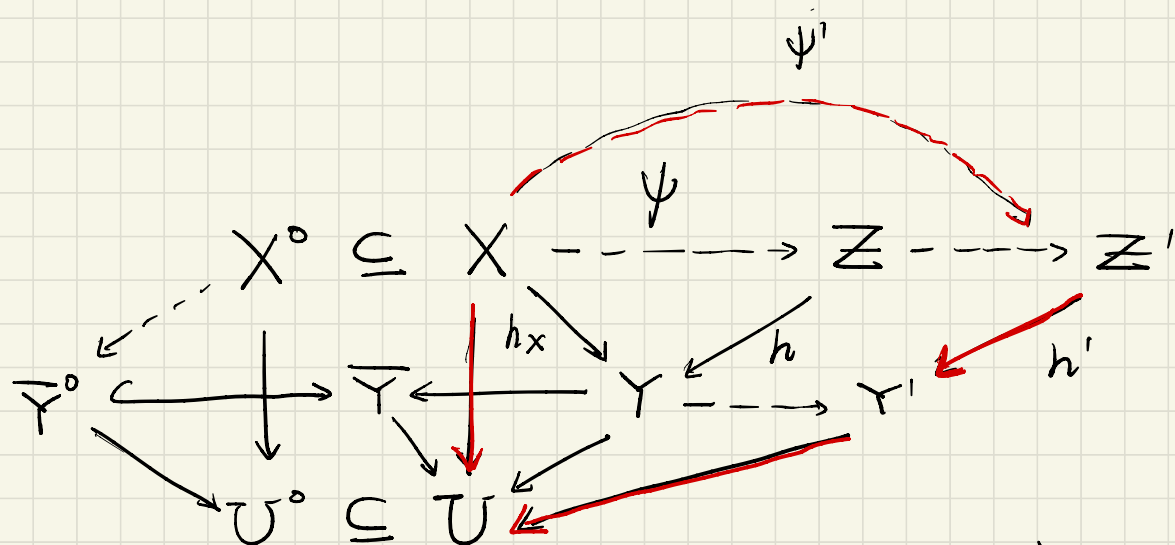
$$(Y, B + J)$$

$$(Y, B + J + \varepsilon(A + C))$$

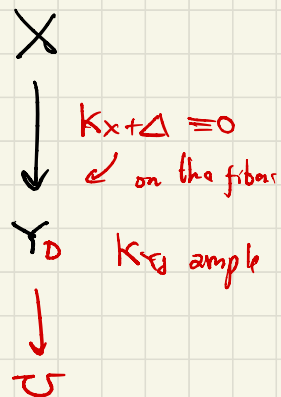
$$S_Q$$

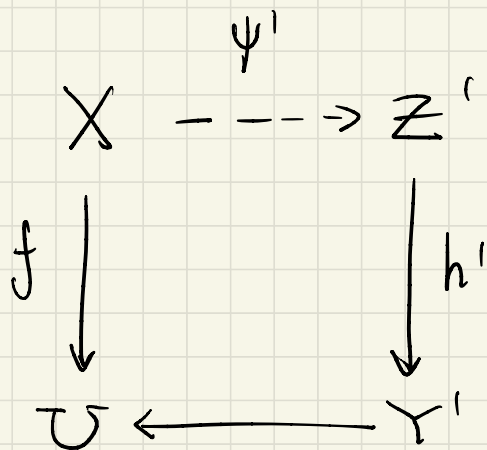
$$(Y, \Theta_\varepsilon).$$

Outcome of 5.2 + 5.3:



$K_X + \Delta$ is ~~not~~ nef over $U \implies$ ample model
 semiample





- $K_{Y'} + B' + J' + t(C' + A')$ semiample over U^o for every $0 \leq t \leq \varepsilon$
- $K_{Y'} + B' + J' + \varepsilon(C' + A')$ semiample over U .
- $K_X + \Delta + \varepsilon h_X^*(C + A)$ has a min model Z' over U .

Aim: produce a min model for $K_{Y'} + B' + J'$ over U .

"Natural": Run a $(K_{Y'} + B' + J')$ -MMP with scaling of $C' + A'$ over U .

Lemma 5.1: We can find a proj bir morphism

$Y \longrightarrow \bar{Y}$ a morphism $h_X: X \longrightarrow Y$:

- (1) (X, Δ) has a good minimal model Z over Y ,
- (2) $h_{X*}(\mathcal{O}_X(m(K_X + \Delta))) \simeq \mathcal{O}_Y(m(K_Y + B + J))$
- (3) $K_Z + \psi_* \Delta \sim_{\mathbb{Q}} h^*(K_Y + B + J)$
- (4) (Y, B) is log smooth and all strata of $T = [B]$ intersects Y° .
- (5) $K_Y + B + J \sim_{\mathbb{Q}, \sigma} C + \Sigma + A$, $(Y, B + \epsilon C)$, $(X, \Delta + \epsilon h_X^* C)$ are dlt
- (6) for any $0 < \epsilon \ll 1$, there exists $\Theta_\epsilon \sim_{\mathbb{Q}, \sigma} B + J + \epsilon(A + C)$
so that (Y, Θ_ϵ) is klt

Proof of 5.1: Proposition 2.1 $(h_X: X \rightarrow Y, B, J)$

$$K_Y + B + J \sim_{\mathcal{O}_Y} C + \Sigma' + A. \quad T = [B]$$

A ample, Σ' & C have no common comp, $\text{Supp}(\Sigma') \subseteq \text{Supp}(T)$

Assume $(Y, \text{supp}(B+C))$ is log smooth

C contains no non-klt centers of $(Y, B) \implies$

$h_X^* C$ contains no non-klt centers of (X, Δ) .

Observe $J + \varepsilon A$ ample over U .

Hence, $B + J + \varepsilon(C + A) \sim_{\mathcal{O}_Y} \Theta_\varepsilon$ where (Y, Θ_ε) is klt.

$$\underbrace{B - \delta T + \varepsilon C}_{\text{klt}} + \underbrace{J + \varepsilon A + \delta T}_{\text{ample}} = B + J + \varepsilon(C + A).$$

Proposition 2.13, gives us an isomorphism:

$$R(X/Y, \mathcal{O}_X(K_X + \Delta + \varepsilon h_X^*(C + A))) \cong$$

$$R(Y/Y, \mathcal{O}_Y(K_Y + B + J + \varepsilon(C + A))) \cong$$

$$R(Y/Y, \mathcal{O}_Y(K_Y + \Theta_\varepsilon)) \leftarrow \text{is f.g.}$$

By proposition 2.11 + 2.12. $\implies (X, \Delta)$ has a
good minimal model (Z, Δ_Z) for (X, Δ) over Y

As an outcome, we conclude that

$$K_Z + \Delta_Z \sim_{\mathbb{Q}, \mathbb{R}} 0 \quad \text{so} \quad K_Z + \Delta_Z \sim_{\mathbb{Q}} h^*(K_Y + B + J).$$

□

Lemma 5.2 + 5.3: Up to replacing X with a higher model:

(1) $K_X + B + J + \varepsilon(C + A)$ has a gmm Y' over U .

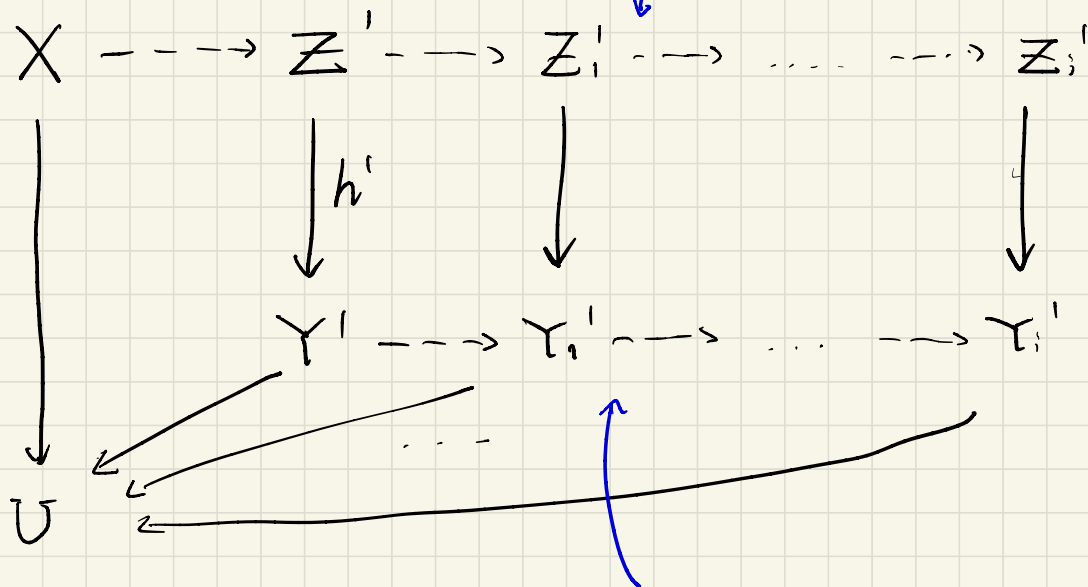
(2) $K_X + \Delta + \varepsilon h_X^*(C + A)$ has a gmm Z' over U ,
equipped with a morphism $Z' \xrightarrow{h'} Y'$.

(3) $(K_{Y'} + B' + J' + t(C' + A'))|_{Y^0}$ is semiample over U^0
for every $0 \leq t \leq \varepsilon$.

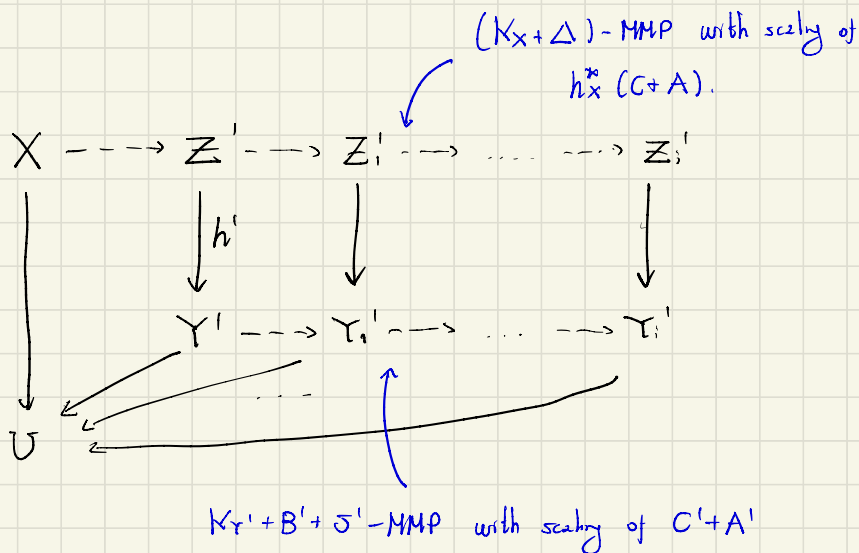
(4) $N_{\sigma}(K_X + B + J + \varepsilon(C + A)/U)$ stabilize for ε small

Prop 2.13.

$(K_X + \Delta)$ -MMP with scaling of $h_X^*(C+A)$.



$K_{Y'} + B' + J'$ -MMP with scaling of $C' + A'$



Claim: This MMP terminates.

$Y' \dashrightarrow Y'_N$, we obtain $Z \dashrightarrow Z'_N$ a min model for $K_Z + \Delta_Z + th^*(C+A)$ and

$$R(X/U, K_X + \Delta) \cong R(Z'_N/U, K_{Z'_N} + \Delta_{Z'_N}).$$

$$R(Z'_N/U^\circ, K_{Z'_N} + \Delta_{Z'_N}) \cong R(X^\circ/U^\circ, K_{X^\circ} + \Delta^\circ) \text{ is f.g.}$$

By induction on dimension, $K_{T'_N} + \Delta_{T'_N} = (K_{Z'_N} + \Delta_{Z'_N})|_{T'_N}$ is semiample over U .

4.1 $\Rightarrow K_{Z'_N} + \Delta_{Z'_N}$ is semiample over U . $\square$

Termination of flips:

Notation: D_i' push-forward of D in Y_i' .

$(K_{Y_i'} + B_i' + \delta_i')$ - MMP over U with scaling of $\varepsilon(C_i' + A_i')$

$$T_i = LB_i' \cdot I_i \text{ for every } i.$$

I_i is a flipping curve. $(C_i' + A_i') \cdot I_i > 0$

$$(K_{Y_i'} + \delta_i' + B_i') \cdot I_i < 0$$

$$\text{so } \sum_i I_i \cdot I_i < 0.$$

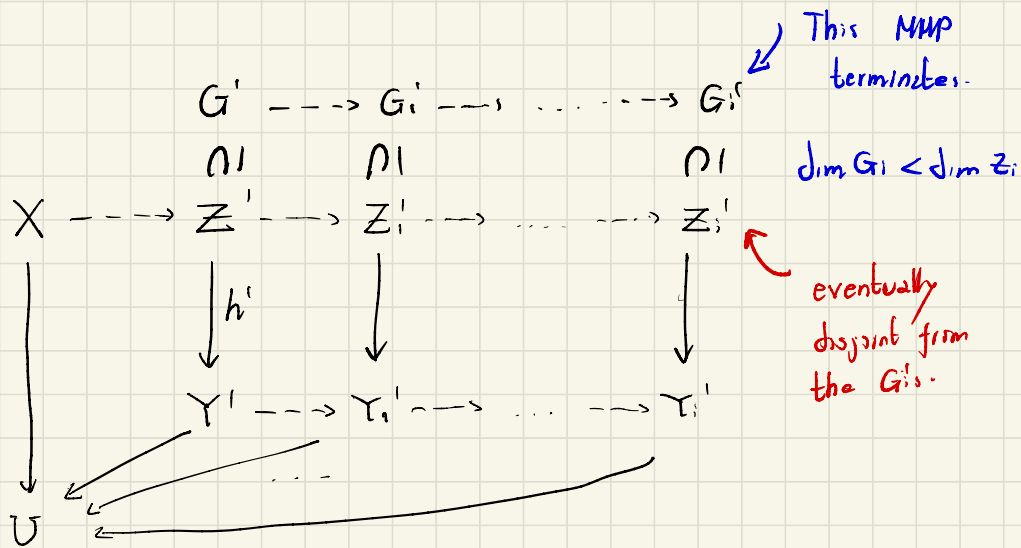
$$\text{supp}(\sum_i I_i) = \text{supp}(LB_i') = \text{supp}(T_i)$$

The whole MMP happens along non-klt (Y_i', B_i') .

G_i' a component of $S = L\Delta_i$ dominating a component of T

We obtain a MMP $G_i' \dashrightarrow G_{i+1}'$ for the

pair (G_i, Δ_{G_i}) obtained by adjunction with scaling of $\Phi_i = h_i^*(A_i' + C_i')|_{G_i}$.



By Thm 1.1 dim $n-1$, $K_{G_i} + \Delta_{G_i}$ has a gmm $\bar{G}$ over U .

$\chi: \bar{G} \longrightarrow \bar{D}$ ample model over U .

Lemma 5.7: There is a gmm $\bar{G}^m$ of $(\bar{G}, \Delta_{\bar{G}} + t_0 \bar{\Phi})$ over $\bar{D}$.

Proposition: $(\bar{G}_m, \Delta_{\bar{G}_m} + t \bar{\Phi}_m)$ is a gmm over U for all $0 \leq t \leq t_0$. Hence, the multi-graded ring

$RC_{\bar{G}_m}/U, K_{\bar{G}_m} + \Delta_{\bar{G}_m}; K_{\bar{G}_m} + \Delta_{\bar{G}_m} + t_0 \bar{\Phi}_m)$ is f.g.

Since $\bar{G} \dashrightarrow \bar{G}_m$ is $K_{\bar{G}} + \Delta_{\bar{G}} + t \bar{\Phi}$ non-pas

$RC_{\bar{G}}/U, K_{\bar{G}} + \Delta_{\bar{G}}; K_{\bar{G}} + \Delta_{\bar{G}} + t_0 \bar{\Phi})$ is also f.g.

Lemma 5.8: The $Z_i \rightarrow Z_{i+1}$ are eventually disjoint from G_i .

Proof: I geom val over $\bar{G}$. $\sigma_D(D) = \inf \{ \text{mult}_D(D') \mid D' \sim D \}$

$$\sigma_D(K_{\bar{G}^m} + \Delta_{\bar{G}^m} + t \bar{\Phi}^m / U) = 0 \text{ for all } t \in [0, b]$$

$\sigma_D(\cdot / U)$ is ≥ 0 and linear on the cone $\mathcal{C}_i \subseteq \text{Div}(G_i)$ spanned by $\gamma_i^*(K_{G_i} + \Delta_{G_i})$ and $\gamma_i^*(K_{G_i} + \Delta_{G_i} + b \Phi_i)$. [CL10].

There exists E over G_i' s.t. for every $0 < s < s_i$ ↗ scaling multiple

$$\alpha(E_i, G_i, K_{G_i} + \Delta_{G_i} + (s_i - s) \Phi_i) <$$

$$\alpha(E_i, G_{i+1}, K_{G_{i+1}} + \Delta_{G_{i+1}} + (s_{i+1} - s) \Phi_{i+1}).$$

This implies $\sigma_{E_i}(K_{G_i} + \Delta_{G_i} + (s_i - s) \Phi_i / U) > 0$.

$$\sigma_E(K_{G_i} + \Delta_{G_i} + s_i \Phi_i / U) = 0.$$

→ ← σ_E is linear and non-neg on $\mathcal{C}_i$.

□